On the Uniform Ergodic Theorem

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Outline of the talk

We shall discuss the following in the lecture.

- We recall some results on weak and weak* topologies.
- Three convergence of bounded linear operators on normed spaces.
- Some ergodic theorems.

Notation

Let us fix some notations.

If A is a subset of a linear space, co A or co(A) is the **convex hull** of A:

$$\left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in A, 0 \leq \alpha_i, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}.$$

We write $A + B$ for $\{x + y : x \in A, y \in B\}$; if A and B are linear spaces and $A \cap B = \{0\}$, we say that $A + B$ is a **direct sum** and write $A \oplus B$. This means the representation $x + y$ is unique.

Notation

Recall that, for a space Y of linear functionals on a space X, the $\sigma(X, Y)$-topology is the coarest topology on X with respect to which all functionals in Y are continuous.

The **weak topology** is the $\sigma(X, X^*)$-topology, where X^* is the space of continuous linear functionals on X.

The ω^*-**topology** is the $\sigma(X^*, X)$-topology on X^*.

lim denotes the limit in the strong (=norm) topology, ω-lim in the weak topology and ω^*-lim in the ω^*-topology.

Notation

- cl A is the closure of $A \subset X$, and ω-cl A is the closure in the weak topology. As usual, $\overline{co}$ A denotes cl co A. Note that this agrees with ω–$\overline{co}$ $A = \omega$–cl co A.
- We denote the space of bounded linear operators in X by $\mathcal{B}(X)$, and $T^\circ = \mathbb{I}$ is the identity operator.
- $\langle x, h\rangle$ is the value of the functional $h \in x^*$ in $x \in X$. That is,

$$\langle x, h\rangle := h(x).$$

The **adjoint or dual operator** of T is the operator $T^* : X^* \to X^*$ with

$$\langle Tx, h\rangle = \langle x, T^* h\rangle \quad \text{for} \quad x \in X, h \in X^*.$$

Weak Topology

Weak topology on a set X using a topological space Y and a collection of functions defined on X :

X is any set and (Y, τ) is a topological space. $\mathtt{F}$ is a family of maps from X into Y. The **weak topology** on X generated by $\mathtt{F}$ (or the $\mathtt{F}$-weak topology) is the weakest (or, the smallest, the coarsest) topology on X for which all $f \in \mathtt{F}$ are continuous. The collection $\{\cap_{j=1}^{k} f_j^{-1}(U_j) : U_j \in \tau, f_j \in \mathtt{F}, 1 \leq j \leq k, k = 1, 2, \ldots\}$ is a base for this topology, or, the collection $\{f_j^{-1}(U_j) : U_j \in \tau, f_j \in \mathtt{F}\}$ is a subbase for this topology.

The topology on a normed space X given by its norm is the **norm/strong/usual topology** on X. The norm topology is "very rich": it has "too many" open sets.

Weak Topology

The weak topology (w-topology) on a normed space X is the weakest topology on X with respect to which all the functions in X^* remain continuous.

To emphasize that it is the weak topology where the elements of X^* act on X as continuous maps, one denotes it by $\sigma(X, X^*)$-topology.

The weak topology exists !

Indeed, the class of topologies for which the elements of X^* act on X as continuous maps is a non-empty class: it certainly includes the norm topology. Taking the intersection of all these topologies, one obtains the weak topology.

Neighborhood Base

A subset of X is open in the weak topology iff it can be written as a union of (possibly infinite many) sets, each of which being an intersection of finitely many sets of the form $f^{-1}(U)$, where U is open in $\mathbb{K}$.

Therefore every weakly open set is strongly open. The norm topology on a normed space X is stronger than the weak topology.

Theorem 1 (Characterization of weak convergence of sequence in X).

$x_n \to x$ (weakly) in the weak topology iff $f(x_n) \to f(x)$, for all $f \in X^*$.

Strong convergence implies weak convergence. The converse is not always true.

But if X is a finite dimensional normed space, then its weak topology is the same as the norm topology.

Weak Topology and Weak Convergence

Proposition 2.

Let X be a normed space. Then

1. *The weak topology on X is a Hausdorff space (a consequence of Hahn-Banach theorem).*
2. *If $x_n \to_w x$ and $y_n \to_w y$, then $x_n + y_n \to_w x + y$ and $\alpha x_n \to_w \alpha x$. Also the weak limit is unique.*
3. *If $x_n \to_w x$, then every subsequence of (x_n) converges weakly to x.*
4. *Every weakly convergent sequence (x_n) is bounded. (a consequence of Uniform Boundedness theorem).*
   An unbounded sequence cannot be a weakly convergent sequence. The analogous statement for generic nets is false. That is, every weakly convergent net is not necessarily bounded.

Theorem 3 (Mazur's Theorem).

Every closed convex subset of X is weakly closed.

Weak Topology and Weak Convergence

The subtlety (not very noticeable) is that **weak topology is not induced from a metric.** Then we must be careful to define convergence in terms of nets rather than just sequences. **Weak convergence of a net in ℓ_1 does not imply norm convergence.** This is usually just a technicality, so often restrict our attention to convergence of sequences. However, it is good to be aware of the distinction.

Theorem 4.

If X is reflexive, then every bounded sequence has a weakly convergent subsequence.

Weak*-Topology

Theorem 5.

Let X be a normed space. For each $x \in X$, consider the evaluation functional ψ_x defined on X^*, $\psi_x(f) = f(x)$. Then ψ_x is a continuous linear functional on X^* since $|\psi_x(f)| = |f(x)| \leq \|x\| \, \|f\|$ for all $f \in X^*$, so that $\psi_x \in X^{**}$ and $\|\psi_x\| \leq \|x\|$. Thus $J : X \to X^{**}$ defined by $x \mapsto \psi_x$ is a bounded linear operator. In fact, J is an isometry.

The weak star topology on X^* is the weakest topology on X^* in which all the functionals $\psi_x : X^* \to \mathbb{K}$ are continuous. To emphasize that it is the weakest topology where the elements of X act on X^* as continuous maps, denoted by $\sigma(X^*, X)$-topology.

Weak*-Topology

1. If X is separable, then every bounded sequence in X^* has a weak* convergent subsequence.
2. The w^*-topology is Hausdorff (Hausdorffness is used for uniqueness).
3. $f_n \to f$ in the weak*-topology iff (f_n) converges to f pointwise, $f_n(x) \to f(x)$, for all $x \in X$ (**characterization of w^*-convergence of sequence in X^***).
4. If X is finite dimensional, the weak*-topology of X^* coincides with the norm topology of X^*.

For infinite dimensional spaces, the w^*-topology never coincides with the norm topology of X^*.

Theorem 6.

Every weak* convergent sequence is bounded. That is, every weak* convergent sequence is uniformly bounded in norm.

Convergence of Sequences of Operators

Let X and Y be normed spaces.

For sequences of operators $T_n \in \mathcal{B}(X, Y)$ there types of convergence turn out be of theoretical as well as practical value. These are

1. Convergence in the norm on $\mathcal{B}(X, Y)$;
2. Strong convergence of $(T_n x)$ in Y ;
3. Weak convergence of $(T_n x)$ in Y.

The definitions and terminology were introduced by J. von Neumann.

Uniform Operator Topology

Definition 7.

Let X and Y be normed spaces. The topology given by the operator norm is called the ***usual topology****, or, the* ***uniform operator topology*** *on $\mathcal{B}(X, Y)$.*

A sequence (T_n) ***converges uniformly*** *to T in the uniform operator topology if $\|T_n - T\| \to 0$. We denote it by $T_n \to T$ and T is called the* ***uniform limit*** *of (T_n). If T is the uniform limit of (T_n), then $\|T\| = \lim \|T_n\|$.*

Strong Operator Topology

Definition 8.

*The **strong operator topology** is the weak topology generated by the family of maps $F_x : \mathcal{B}(X, Y) \to Y$ by $T \mapsto Tx$, where x varies over X.*

*A net T_α in $\mathcal{B}(X, Y)$ **converges strongly** to T in the **strong operator topology** if $\|T_\alpha x - Tx\| \to 0$ for each $x \in X$.*

*We denote it by $T_\alpha \to_s T$ and T is called the **strong limit** of (T_α).*

Convergence of Sequences of Operators

Definition 9.

*The **weak operator topology** is the weak topology generated by the family of maps $F_{x,y^*} : \mathcal{B}(X, Y) \to \mathbb{K}$ by $T \mapsto y^*(Tx)$, where x varies over X and y^* varies over Y^*.*

*A net T_α in $\mathcal{B}(X, Y)$ converges weakly to T in the weak operator topology if $y^*(T_\alpha x)$ converges to $y^*(Tx)$, for all $y^* \in Y^*$, $x \in X$. We denote it by $T_\alpha \to_w T$. T is called the **weak limit** of (T_α). Neigbourhood base at $0 = \{T : |y^*(Tx_i)| < \varepsilon, i = 1, 2, \ldots, n\}$.*

It is not difficult to show that uniform convergent $\Rightarrow$ strong convergent $\Rightarrow$ weak convergent (the limit being the same), but the converse is not generally true, as can be seen from the following examples. If X is a finite dimensional normed space, then all convergences are same.

If $T_n \in \mathcal{B}(X, Y)$ and $T_n \to T$ uniformly, then T need not be bounded. If X is Banach, then T is always bounded.

Theorem 10 (Banach-Steinhauss Theorem, 1927).

Let X be a Banach space, Y be a normed space and (T_n) be a sequence in $\mathcal{B}(X, Y)$ converging to T (pointwise). Then $(\|T_n\|)$ is a bounded sequence and $T \in \mathcal{B}(X, Y)$.

Moreover, this theorem does not say that $T_n \to T$ uniformly. But it says that if $(T_n x)$ converges strongly for each $x \in X$, then (T_n) converges strongly to some $T \in \mathcal{B}(X, Y)$.

The completeness hypothesis on X cannot be dropped from Banach-Steinhauss Theorem.

Theorem 11.

Let (f_n) be a sequence of bounded linear functionals on a Banach space X. Suppose for each x, $(f_n(x))$ converges to a limit $f(x)$. Then f is a bounded linear functional.

This result says that the point limit of continuous linear functionals is continuous linear. But the result is not true if we remove linearity.

Example 12.

$f_n : [0,1] \to [0,1]$ *defined by* $f_n(x) = x^n$. *Each* f_n *is continuous and* $f_n(x) \to f(x)$ *but* f *is discontinuous.*

Theorem 13.

Let X be a Banach space and Y, Z be normed spaces. Let A_n, A be linear operators from X to Y and B_n, B be linear operators from Z to X. If each A_n is continuous, $A_n x \to Ax$ for every $x \in X$ and $B_n z \to Bz$ for every $z \in Z$, then $A_n B_n z \to ABz$ for every $z \in Z$. That is, if $A_n \to_s A$ and $B_n \to_s B$, then $B_n A_n \to_s BA$.

Convergence of Sequences of Operators

If the convergence is uniform, then $T \in \mathcal{B}(X, Y)$. If the convergence is strong or weak, T is still linear but may be unbounded if X is not complete.

Example 14.

Consider X as c_{00} with respect to $\|.\|_2$. A sequence of bounded linear operators T_n on X is defined by

$$T_n x = (x_1, 2x_2, 3x_3, \ldots, nx_n, x_{n+1}, x_{n+2}, \ldots).$$

This sequence (T_n) converges strongly to the unbounded operator T defined by $Tx = (x_1, 2x_2, 3x_3, \ldots)$. However, if X is complete, the situation illustrated by this example cannot occur for strong operator convergence.

Theorem 15 (Uniform Boundedness Theorem).

Let X and Y be normed spaces and (T_n) be a sequence in $\mathcal{B}(X,Y)$ such that $(T_n x)$ converges in Y for every $x \in X$. If $(\|T_n\|)$ is a bounded sequence, then the operator $T : X \to Y$ defined by $Tx = \lim_n T_n x$, $x \in X$, belongs to $\mathcal{B}(X,Y)$ and $\|T\| \leq \liminf_n \|T_n\|$.

By Uniform Boundedness theorem, the condition of boundedness of $(\|T_n\|)$ is redundant if X is Banach. That is, if X is Banach and if $T_n \to_s T$, then T is also bounded by Uniform Boundedness principle.

What is the relation between this T and the uniform limit of (T_n)?

From above point, the norm of T is less than or equal to the norm of uniform limit of (T_n).

Convergence of Sequences of Operators

Suppose X is Banach. If a sequence (T_n) is Cauchy in the strong sense, that is, for all $x \in X$ the sequence $(T_n x)$ is Cauchy in X, then there exists $T \in \mathcal{B}(X)$ such that $T_n \to T$ strongly. That is, if X is complete, then the strong operator topology on $\mathcal{B}(X, Y)$ is complete.

The sequence (T_n) in $\mathcal{B}(X, Y)$ is said to be a strong Cauchy sequence if the sequence $(T_n x)$ is a Cauchy sequence for all $x \in X$. The above result says that **if the spaces X and Y are Banach spaces, then $\mathcal{B}(X, Y)$ is complete in the strong sense.**

Proposition 16.

If X is finite dimensional and (T_n) is a sequence in $\mathcal{B}(X, Y)$ such that $T_n x \to Tx$ for all $x \in X$. Then $T \in \mathcal{B}(X, Y)$ and $\|T_n - T\| \to 0$.

If X is finite dimensional, then the uniform operator topology in $\mathcal{B}(X, Y)$ is complete.

Definition 17.

Let X be a Banach space and $T \in \mathcal{B}(X)$. The operator T is called ***power bounded*** *if the norms of the powers T^n $(n \geq 0)$ are uniformly bounded* $(\sup_n \|T^n\| < \infty)$ *;*

and ***Cesàro bounded*** *if the norms of the* ***Cesàro averages*** *of T*

$$A_n(T) := \frac{\mathbb{I} + T + \cdots + T^{n-1}}{n}$$

are uniformly bounded.

Uniformly/mean/weakly mean ergodic operator

Definition 18.

Let X be a Banach space and $T \in \mathcal{B}(X)$.

1. *If $\{A_n(T)\}$ converges uniformly in $\mathcal{B}(X)$, then T is called* ***uniformly ergodic****.*
2. *If $\{A_n(T)\}$ converges strongly in $\mathcal{B}(X)$, then T is called* ***mean ergodic****.*
3. *If $\{A_n(T)\}$ converges weakly in $\mathcal{B}(X)$, then T is called* ***weakly mean ergodic****.*

As "**uniform convergent** $\implies$ **strong convergent** $\implies$ **weak convergent**" (the limit being the same), we have T is uniformly ergodic $\implies$ mean ergodic $\implies$ weakly mean ergodic.

Uniformly/mean/weakly mean ergodic operator

The following identity is useful to say the convergence of the sequence $\left\{\frac{T^{n-1}}{n}\right\}_{n\geq 1}$ when T is uniformly/mean/weakly mean ergodic :

$$T^{n-1} = nA_n(T) - (n-1)A_{n-1}(T).$$

So,

$$\frac{T^{n-1}}{n} = A_n(T) - \frac{(n-1)}{n}A_{n-1}(T).$$

- If T is power bounded, then T is Cesàro bounded.
- If T is Cesàro bounded, then $\left\{\frac{T^{n-1}}{n}\right\}$ is uniformly bounded.
- If T is uniformly ergodic, then T is Cesàro bounded.
- If T is uniformly ergodic, then $\left\{\frac{T^{n-1}}{n}\right\}$ is uniformly bounded.

Uniformly/mean/weakly mean ergodic operator

- If T is power bounded, then $\left\{\frac{T^{n-1}}{n}\right\} \to 0$ uniformly.
- If T is power bounded, then T is Cesàro bounded and $\left\{\frac{T^{n-1}}{n}\right\} \to 0$ uniformly.
- If T is mean ergodic, then T is Cesàro bounded and $\left\{\frac{T^{n-1}}{n}\right\} \to 0$ strongly.
- If T is weakly mean ergodic, then T is Cesàro bounded and $\left\{\frac{T^{n-1}}{n}\right\} \to 0$ weakly.

von Neumann's Mean Ergodic Theorem, 1931

Theorem 19.

Let U be a unitary operator on a Hilbert space (more generally, an isometric linear operator,

$$\|Ux\| = \|x\|, \quad \text{for all } x \in H,$$

not necessarily surjective, that is, $UU^* = \mathbb{I}$, but not necessarily $UU^* = \mathbb{I}$). The sequence of averages $\{A_n(U)\}$ converges to P in the strong operator topology, where P is the orthogonal projection onto the closed subspace $N(\mathbb{I} - U)$.

That is, every unitary operator (not necessarily surjective) is mean ergodic.

Mean ergodic theorems for operators

Von Neumann [1931]	proved for unitary operators in a complex Hilbert space
Visser	proved for power bounded operators on a Hilbert space
Riesz	proved for power bounded operators on $L_p, \ 1 < p < \infty$
Lorch, Kaukutani Yosida (independently)	proved for power bounded operators in a reflexive (real or complex) Banach space

Examples

Hile	gave examples of mean ergodic operators which are not power bounded
Derriennic	gave an example of a mean ergodic operator T in a Hilbert space for which $\frac{\|T^n\|}{n} \nrightarrow 0$, and T^* is not mean ergodic (but only weakly mean ergodic)
Tomilov and Zemanek	gave a general method of constructing such examples
Lorch, Kaukutani Yosida (independently)	proved for power bounded operators in a reflexive (real or complex) Banach space
ToZ	provided an example that both T and T^* weakly mean ergodic but not mean ergodic
Kornfeld and Keosk	constructed for every $\varepsilon > 0$, a mean ergodic positive operator T on L_1 such that $\frac{\|T^n\|}{n^{1-\varepsilon}} \to \infty$; by Cesàro boundedness, $\frac{\|T^n\|}{n}$ is bounded.

Theorem 20 (U. Krengel, p.72).

Let T be a Cesàro bounded linear operator in a Banach space X. For any $x \in X$, satisfying $\lim_{n\to\infty} \frac{T^{n-1}}{n}x = 0$, and any $y \in X$ the following assertions are equivalent :

1. $Ty = y$ *and* $y \in \overline{co}\{x, Tx, T^2x, \cdots\}$;
2. $y = \lim A_n x$;
3. $y = w\text{-}\lim A_n x$;
4. y *is a weak cluster point of the sequence* $\{A_n x\}$.

Corollary 21.

Let T be a power bounded operator on X. Then T is mean ergodic if and only if T is weakly mean ergodic.

Mean Ergodic Theorem

The present formulation of the mean ergodic theorem is a special ease of results of Eberlein [1949], but the main assertions emerged already with the work of F.Riesz [1938] for $X = L_p$, and independently with the work of K. Yosida [1938] and S. Kakutani [1938] for general Banach spaces; see also Yosida-Kakutani [1941].

The following important special case was proved by E.Lorch [1939] independently:

Theorem 22.

If T is a power bounded linear operator in a reflexive Banach space X, the averages $\{A_n x\}$ converges in norm to a T-invariant limit for all $x \in X$.

Our next aim is a **splitting theorem** for

$$X_{me} = X_{me}(T) = \{x \in X : \lim A_n x \text{ exists}\}.$$

Clearly, if T is Cesàro bounded, X_{me} is a closed linear subspace of X.

T is called **mean ergodic** if $X = X_{me}$.

We shall use the notation

$$\begin{aligned} F = F(T) &:= \{x \in X : Tx = x\}, \\ N &:= \{x - Tx : x \in X\} = (\mathbb{I} - T)X, \\ F_* = F_*(T) &:= \{h \in X^* : T^* h = h\}, \\ N_* &:= (\mathbb{I} - T^*)X^*. \end{aligned}$$

Splitting Theorem

Most of the next theorem is due to Yosida [1938] :

Theorem 23.

Let T be Cesàro bounded, and assume that $\lim \frac{T^{n-1}}{n} x = 0$ *holds for all* $x \in X$. *Then* $X_{me} = F \oplus cl\, N$. *The operator P assigning to* $x \in X_{me}$ *the limit* $Px := \lim A_n x$ *is the projection of* X_{me} *onto F.*

We have $P = P^2 = TP = PT$.

For any $z \in X$, *the following assertions are equivalent :*

1. $\lim A_n z = 0$;
2. $\langle z, h \rangle = 0$ *for all* $h \in F_*$;
3. $z \in cl\, N$.

are equivalent.

Recall that a linear operator Q is called **projection** if $Q = Q^2$.

A projection with $Q = QT = TQ$ will be called T-**absorbing**. Thus, P is a T-absorbing projection.

Occasionally, the following criterion of Sine [1970] for mean ergodicity is useful:

Theorem 24.

Let T be Cesàro bounded and $\lim \frac{T^{n-1}}{n}x = 0$ holds for all x. Then T is mean ergodic iff F separates F_*.

References

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